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Strong Efficient Edge Domination number of some graphs obtained by duplicating their elements

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Abstract

Let $G = (V, E)$ be a simple graph. A subset S of E(G) is a strong (weak) efficient edge dominating set of G if $|N_s[e] \cap S| = 1$ for all $e \in E(G)$($|N_w[e] \cap S| = 1$ for all $e \in E(G)$) where $N_s(e) = \{f / f \in E(G)$ & $\deg f \geq \deg e\}$($N_w(e) = \{f / f \in E(G)$ & $\deg f \leq \deg e\}$) and $N_s[e] = N_s(e) \cup \{e\}$($N_w[e] = N_w(e) \cup \{e\}$). The minimum cardinality of a strong efficient edge dominating set of G (weak efficient edge dominating set of G) is called a strong efficient edge domination number of G and is denoted by $\gamma'_{se}(G)$ ($\gamma'_{we}(G)$). In this paper, the strong efficient edge domination number of some graphs obtained by duplicating their elements is studied.

Keywords: Domination, edge domination, strong edge domination, efficient edge domination, strong efficient edge domination.

1. INTRODUCTION

Throughout this paper, only finite, undirected and simple graphs are considered. Two volumes on domination have been published by T. W. Haynes, S. T. Hedetniemi and P. J. Slater [9, 10]. Edge dominating sets were studied by S. L. Mitchell and S. T. Hedetniemi [12]. A set F of edges in a graph G is called an edge dominating set of G if every edge in E – S is adjacent to at least one edge in F. The edge domination number $\gamma'(G)$ of a graph G is the minimum cardinality of an edge dominating set of G. The degree of an edge was introduced by V. R. Kulli [8]. The concept of efficient domination was introduced by D.W. Bange et al [4, 5]. The concept of strong domination graphs was introduced by E. Sampath Kumar and L. Pushpalatha [13] and efficient edge domination were studied by C. L. Lu et al [11] G. Santhosh [14] and D. M. Cardoso et al [6]. The strong efficient edge domination was introduced by M. Annapoopathi and N. Meena [1,2,3]. For all graph theoretic terminologies and notations, Harary [7] is referred to. The strong (weak) domination number $\gamma'_s(G)$($\gamma'_w(G)$) of G is the minimum cardinality of a strong (weak) dominating set of G and $\Gamma_s(G)$ is the maximum cardinality of a minimal strong dominating set of G. A subset D of E (G) is called an efficient edge dominating set if every edge in E (G) is dominated by exactly one edge in D. The cardinality of the minimum efficient edge dominating set is called the efficient edge domination number of G. In this paper, the strong efficient edge domination numbers of some graphs obtained by duplicating their elements is studied.

Definition 1.1[1] : Let $G = (V, E)$ be a simple graph. A subset S of E(G) is a strong (weak) efficient edge dominating set of G if $|N_s[e] \cap S| = 1$ for all $e \in E(G)$ [$|N_w[e] \cap S| = 1$ for all $e \in E(G)$] where $N_s(e) =$

$\{f/f \in E(G) \& deg f \geq \deg e\}$ ($N_w(e) = \{f/f \in E(G) \& deg f \leq \deg e\}$) and $N_s[e] = N_s(e) \cup \{e\} (N_w[e] = N_w(e) \cup \{e\})$. The minimum cardinality of a strong efficient edge dominating set of G (weak efficient edge dominating set of G) is called a strong(weak) efficient edge domination number of G and is denoted by $\gamma'_{se}(G)$ ($\gamma'_{we}(G)$).

Observation 1.2[1]:$\gamma'_{se}(C_{3n}) = n, \ \forall n \in N$.

Definition 1.3: Duplication of avertex v of a graph G produces a new graph G' by adding a new vertex v' such that $N[v'] = N[v]$. In other words a vertex v' is said to be a duplication of v if all vertices which are adjacent to v in G are also adjacent to v' in G'.

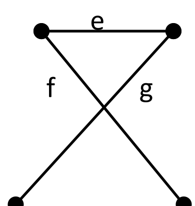
Definition 1.4: Duplication of an edge e = uv by a new vertex w in a graph G produces a new graph G'such that $N[v] = \{u, v\}$.

II. Strong efficient edge domination number of some graphs obtained by duplicating their elements

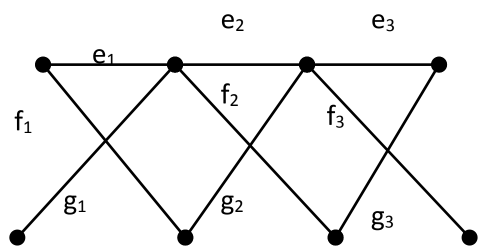
Theorem 2.1: Let $G = P_n, n \geq 2$. Let G' be the graph obtained by duplicating all the vertices of G. Then strong efficient edge dominating set of G' exists if and only if n = 2, 4, 5, 6.

Proof: Let $G = P_n, n \geq 2$. Let G' be the graph obtained by duplicating all the vertices of G.

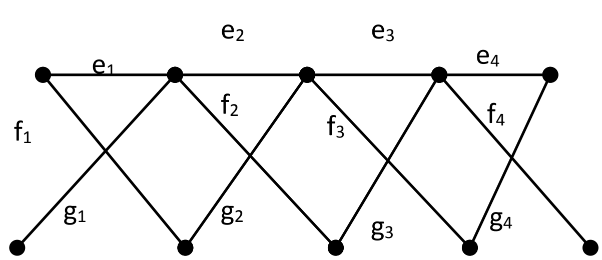
Case (1): Let$G = P_2$. The graph G' is given in the following figure. Then {e} is the unique strong efficient edge dominating set ofG'. Hence $\gamma'_{se}(G') = 1$.



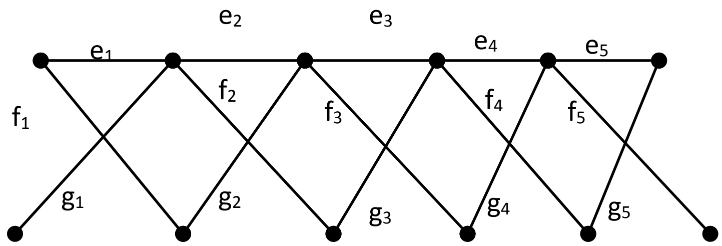
Case (2): Let$G = P_4$. The graph G' is given in the following figure. Then {e_2, f_1, g_3} is the unique strong efficient edge dominating set ofG'. Hence $\gamma'_{se}(G') = 3$.



Case (3): Let$G = P_5$. The graph G' is given in the following figure. Then S_1= {e_2, e_4, f_1}, S_2= {e_1, e_3, g_4}are the strong efficient edge dominating sets of$G'$and $|S_i|$ = 2, i = 1, 2. Therefore $\gamma'_{se}(G') = 3$.



Case (4): Let $G = P_6$. The graph G' is given in the following figure. Then {e_1, e_3, e_5} is the unique strong efficient edge dominating set of G'. Hence $\gamma_{se}^{'}(G') = 3$.



Conversely: Case (1): Let G = P_3. Let V($G^{'}$) = {v_i, u_i / $1 \le i \le 3$}, $e_1 = v_1 v_2$, $e_2 = v_2 v_3$, $f_1 = v_1 u_2$, $f_2 = v_2 u_3$, $g_1 = u_1 v_2$, $g_2 = u_2 v_3$. Then E(G') = {e_i, f_i ,g_i / $1 \le i \le 2$}. Deg e_1 = deg e_2 = 4, deg f_1 = deg g_2 = 2, deg f_2 = deg g_1 = 3. Let S be a strong efficient edge dominating set of G'. Since deg e_1 = deg e_2 = 4 = $\Delta(G^{'})$, any one of the edge e_1 or e_2 belongs to S. Suppose e_1 belongs to S. It strongly efficiently dominates all the edges other than g_2. If the edge g_2 belongs to S, then $|N_S[f_1] \cap S| = |\{e_1, g_2\}| = 2 > 1$, a contradiction. Hence g_2 does not belongs to S. The proof is similar if the edge e_2 belongs to S. Hence $G^{'}$ has no strong efficient edge dominating set.

Case (2): Let $G = P_{3n}$, $n \ge 3$. Let V($G^{'}$) = {v_i, u_i / $1 \le i \le 3n$}, $e_i = v_i v_{i+1}$, $f_i = v_i u_{i+1}$, $g_i = u_i v_{i+1}$, $1 \le i \le 3n - 1$. Then E(G') = {e_i, f_i ,g_i / $1 \le i \le 3n - 1$}. Deg e_1 = deg e_{3n-1} = 4, deg e_i = 6, $2 \le i \le 3n - 2$, deg f_1 = deg g_{3n-1} = 2, deg g_1 = deg f_{3n-1} = 3 and the remaining edges have degree 4. Let S be a strong efficient edge dominating set of G'. Since deg e_i = 6 = $\Delta(G^{'})$, $2 \le i \le 3n - 2$, either the edge e_2 or e_3 belongs to S.

Sub case 2(a): Suppose the edge e_2 belongs to S. Then it strongly efficiently dominate e_1, e_3, f_2, f_3, g_1, g_2. Also the edges e_5, e_8, e_{11},…, e_{3n-4} belong to S. If the edge e_{3n-2} belongs to S then $|N_S[e_{3n-3}] \cap S| = |\{e_{3n-2}, e_{3n-4}\}| = 2 > 1$, a contradiction. Hence there is no edge in S to strongly efficiently dominate e_{3n-2}. Therefore S is not a strong efficient edge dominating set of $G^{'}$. Hence $G^{'}$ has no strong efficient edge dominating set.

Sub case 2(b): Suppose the edge e_3 belongs to S. Then it strongly efficiently dominates e_2, e_4, f_3, f_4, g_2, g_3. Also the edges e_1, e_6, e_9,…, e_{3n-3}, e_{3n-1} belong to S. Also any one of the edges from each of the following set of edges $\{g_4, f_5\}, \{g_7, f_8\}, ..., \{g_{3n-5}, f_{3n-4}\}$ belongs to S. Suppose the edge g_4 belongs to S then $|N_S[f_3] \cap S| = |\{e_3, g_4\}| = 2 > 1$, a contradiction. Suppose the edge f_5 belongs to S then $|N_S[g_6] \cap S| = |\{e_6, f_5\}| = 2 > 1$, a contradiction. Hence there is no edge in S to strongly efficiently dominate g_4 and f_5 Therefore S is not a strong efficient edge dominating set of $G^{'}$. The proof is similar for the remaining set of edges. Hence $G^{'}$ has no strong efficient edge dominating set.

Case (3): Let $G = P_{3n+1}$, $n \ge 2$. Let V($G^{'}$) = {v_i, u_i / $1 \le i \le 3n+1$}, $e_i = v_i v_{i+1}$, $f_i = v_i u_{i+1}$, $g_i = u_i v_{i+1}$, $1 \le i \le 3n$. Then E(G') = {e_i, f_i ,g_i / $1 \le i \le 3n$}. Deg e_1 = deg e_{3n} = 4, deg e_i = 6, $2 \le i \le 3n - 1$, deg f_1 = deg g_{3n} = 2, deg g_1 = deg f_{3n} = 3 and the remaining edges have degree 4. Let S be a strong efficient edge dominating set of G'. Since deg e_i = 6 = $\Delta(G^{'})$, $2 \le i \le 3n - 1$, either the edge e_2 or e_3 belongs to S.

Sub case 3(a): Suppose the edge e_2 belongs to S. Then it strongly efficiently dominate e_1, e_3, f_2, f_3, g_1, g_2. Also the edges e_5, e_8, $e_{11}, \ldots$, e_{3n-1} belongs to S. If the edge g_3 belongs to S then, $|N_S[f_2] \cap S| = |\{e_2, g_3\}| = 2 > 1$, a contradiction. Suppose the edge f_4 belongs to S then $|N_S[g_5] \cap S| = |\{e_5, f_4\}| = 2 > 1$, a contradiction. Hence there is no edge in S to strongly efficiently dominate g_3 and f_4. Therefore S is not a strong efficient edge dominating set of G'. Hence G' has no strong efficient edge dominating set.

Sub case 3(b): Suppose the edge e_3 belongs to S. Then it strongly efficiently dominates e_2, e_4, f_3, f_4, g_2, g_3. Also the edges e_1, e_6, e_9, …, e_{3n-3} belongs to S. Also any one of the edges from each of the following set of edges $\{g_4, f_5\}, \{g_7, f_8\}, \ldots, \{g_{3n-5}, f_{3n-4}\}$ belongs to S. Suppose the edge e_{3n-1} belongs to S then $|N_S[e_{3n-2}] \cap S| = |\{e_{3n-1}, e_{3n-3}\}| = 2 > 1$, a contradiction. Hence there is no edge in S to strongly efficiently dominate e_{3n-1}. The proof is similar for the remaining set of edges. Therefore S is not a strong efficient edge dominating set of G'. Hence G' has no strong efficient edge dominating set.

Case (4): Let $G = P_{3n+2}$, $n \geq 2$. Let $V(G') = \{v_i, u_i \,/\, 1 \leq i \leq 3n+2\}$, $e_i = v_i v_{i+1}$, $f_i = v_i u_{i+1}$, $g_i = u_i v_{i+1}$, $1 \leq i \leq 3n+1$. Then $E(G') = \{e_i, f_i, g_i \,/\, 1 \leq i \leq 3n+1\}$. Deg e_1 = deg e_{3n+1} = 4, deg $e_i = 6$, $2 \leq i \leq 3n$, deg f_1 = deg g_{3n+1} = 2, deg g_1 = deg f_{3n+1} = 3 and the remaining edges have degree 4. Let S be a strong efficient edge dominating set of G'. Since deg $e_i = 6 = \Delta(G')$, $2 \leq i \leq 3n$, either the edge e_2 or e_3 belongs to S.

Sub case 4(a): Suppose the edge e_2 belongs to S. Then it strongly efficiently dominate e_1, e_3, f_2, f_3, g_1, g_2. Also the edges e_5, e_8, $e_{11}, \ldots$, e_{3n-1}, e_{3n+1} belongs to S. If the edge g_3 belongs to S then, $|N_S[f_2] \cap S| = |\{e_2, g_3\}| = 2 > 1$, a contradiction. Suppose the edge f_4 belongs to S then $|N_S[g_5] \cap S| = |\{e_5, f_4\}| = 2 > 1$, a contradiction. Hence there is no edge in S to strongly efficiently dominate g_3 and f_4. Therefore S is not a strong efficient edge dominating set of G'. Hence G' has no strong efficient edge dominating set.

Sub case 4(b): Suppose the edge e_3 belongs to S. Then it strongly efficiently dominates e_2, e_4, f_3, f_4, g_2, g_3. Also the edges e_1, e_6, e_9, …, e_{3n} belongs to S. Also any one of the edges from each of the following set of edges $\{g_4, f_5\}, \{g_7, f_8\}, \ldots, \{g_{3n-2}, f_{3n-1}\}$ belongs to S. If the edge g_4 belongs to S then $|N_S[f_3] \cap S| = |\{e_3, g_4\}| = 2 > 1$, a contradiction. If the edge f_5 belongs to S then $|N_S[g_6] \cap S| = |\{e_6, f_5\}| = 2 > 1$, a contradiction. Hence there is no edge in S to strongly efficiently dominate g_4 and f_5. The proof is similar for the remaining set of edges. Therefore S is not a strong efficient edge dominating set of G'. Hence G' has no strong efficient edge dominating set.

Theorem 2.2: Let $G = K_{1,n}, n \geq 1$. Let G' be the graph obtained by duplicating all the vertices of G. Then $\gamma'_{se}(G') = 1, n \geq 1$.

Proof: Let $G = K_{1,n}, n \geq 1$. Let $V(G') = \{v, v_i, u_i \,/\, 1 \leq i \leq n, n \geq 1\}$ and $E(G') = \{vv_i, vu_i, 1 \leq i \leq n, n \geq 1\}$. Then deg vv_i = deg vu_i = $2n - 1 = \Delta(G')$, $1 \leq i \leq n$. Therefore $\{vv_i / 1 \leq i \leq n\}$ or $\{vu_i / 1 \leq i \leq n\}$ is a strong efficient edge dominating set of G'. Hence $\gamma'_{se}(G') = 1, n \geq 1$.

Theorem 2.3: Let $G = D_{r,s}, r, s \geq 1$. Let G' be the graph obtained by duplicating all the vertices of G. Then $\gamma'_{se}(G') = 1, r, s \geq 1$.

Proof: Let $G = D_{r,s}, r, s \geq 1$. Let V(G') = {u, v, u_i, v_j / $1 \leq i \leq r$, $1 \leq j \leq s, r, s \geq 1$} and E(G') = {$uu_i$, vv_j, uu_i', vv_j', $1 \leq i \leq r$, $1 \leq j \leq s, r, s \geq 1$}. Then deg uv = 2(r+s), deg uu_i = deg uu_i' = 2r, deg vv_j = deg vv_j' = 2s, $r, s \geq 1$ Therefore { uv } is the unique strong efficient edge dominating set of G'. Hence $\gamma_{se}'(G') = 1, r, s \geq 1$.

Theorem 2.4: Let $G = C_{3n}, n \geq 1$. Let G' be the graph obtained by duplicating all the vertices of G. Then G' has no strong efficient edge dominating set.

Proof: Let G = C_{3n}, $n \geq 1$. Let V(G') = {u_i, v_i / $1 \leq i \leq 3n$}, $e_i = v_i v_{i+1}$, $1 \leq i \leq 3n-1$, $e_{3n} = v_{3n} v_1$, $f_i = u_i v_{i+1}$, $1 \leq i \leq 3n-1$, $f_{3n} = u_{3n} v_1$, $g_i = v_i u_{i+1}$, $1 \leq i \leq 3n-1$, $g_{3n} = v_{3n} u_1$. Then E(G') = {e_i, f_i, g_i / $1 \leq i \leq 3n$}. Deg e_i = 6, $1 \leq i \leq 3n$, deg f_i = deg g_i = 4, $1 \leq i \leq 3n$. Let S be a strong efficient edge dominating set of G'. Since deg $e_i = 6 = \Delta(G')$, $1 \leq i \leq 3n$, any one of the edges e_i, $1 \leq i \leq 3n$ belongs to S. Without loss of generality let it be the edge e_1.

Then it strongly efficiently dominates e_2, e_{3n}, f_1, f_{3n}, g_2. Also the edges e_4, e_7, e_{10}, …, e_{3n-2} belongs to S. Also any one of the edges from each of the following set of edges $\{f_2, g_3\}, \{f_5, g_6\}, ..., \{f_{3n-1}, g_{3n}\}$ must belong to S. If the edge f_2 belongs to S then $|N_S[g_1] \cap S| = |\{e_1, f_2\}| = 2 > 1$, a contradiction. If the edge g_3 belongs to S then $|N_S[f_4] \cap S| = |\{e_4, g_3\}| = 2 > 1$, a contradiction. Hence there is no edge in S to strongly efficiently dominate f_2 and g_3. The proof is similar for the remaining set of edges. Therefore S is not a strong efficient edge dominating set of G'. Hence G' has no strong efficient edge dominating set.

Theorem 2.5: Let $G = W_{3n}, n \geq 2$. Let G' be the graph obtained by duplicating all the vertices of G. Then G' has no strong efficient edge dominating set.

Proof: Let G = W_{3n}, $n \geq 2$. Let V(G') = {u, u_i, v_i / $1 \leq i \leq 3n - 1$}, $e_i = uv_i$, $1 \leq i \leq 3n-1$, $e_i' = uu_i$, $1 \leq i \leq 3n-1$, $f_i = v_i v_{i+1}$, $1 \leq i \leq 3n-2$, $f_{3n-1} = v_{3n-1} v_1$, $g_i = u_i v_{i+1}$, $1 \leq i \leq 3n-2$, $g_{3n-1} = u_{3n-1} v_1$, $h_i = v_i u_{i+1}$, $1 \leq i \leq 3n-2$, $h_{3n-1} = v_{3n-1} u_1$. Then E(G') = {e_i, e_i', f_i, g_i, h_i / $1 \leq i \leq 3n$}. Deg e_i = 6n+1, deg e_i' = 6n-1, $1 \leq i \leq 3n-1$, deg f_i = 8, deg g_i = deg h_i = 6, $1 \leq i \leq 3n-1$. Let S be a strong efficient edge dominating set of G'. Since deg $e_i = 6n+1 = \Delta(G')$, $1 \leq i \leq 3n-1$, any one of the edges e_i, $1 \leq i \leq 3n-1$ belongs to S. Without loss of generality let it be the edge e_1. Then it strongly efficiently dominate all the spoke edges e_i', $1 \leq i \leq 3n-1$ and two rim edges adjacent with e_i. Also the edges f_3, f_6, f_9, …, f_{3n} belongs to S. Also any one of the edges from each of the following set of edges $\{h_2, g_1\}, \{h_5, g_4\}, ..., \{h_{3n-1}, g_{3n-2}\}$ must belong to S.

Case(1): Suppose the edge h_2 belongs to S then $|N_S[g_3] \cap S| = |\{h_2, f_3\}| = 2 > 1$, a contradiction.

Case(2): Suppose the edge g_{3n-2} belongs to S then $|N_S[h_{3n-3}] \cap S| = |\{f_{3n}, g_{3n-2}\}| = 2 > 1$, a contradiction.

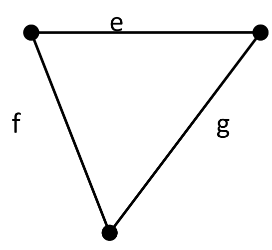
Case(3): Suppose the edge h_{3n-1} belongs to S. Then it strongly efficiently dominates g_1 and g_{3n-2}. But there is no edge to strongly dominate the edge h_2, a contradiction.

Case(4): Suppose the edge g_1 belongs to S. Then it strongly efficiently dominates h_2 and h_{3n-1}. But there is no edge to strongly dominate the edge g_{3n-2}, a contradiction.The proof is similar for the remaining set of edges. Therefore S is not a strong efficient edge dominating set of G'. Hence G' has no strong efficient edge dominating set.

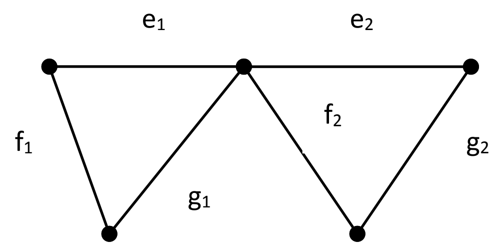
Theorem 2.6: Let $G = P_n, n \geq 2$. Let G' be the graph obtained by duplicating all the edges of G by vertices. Then strong efficient edge dominating set of G' exists if and only if $2 \leq n \leq 6$.

Proof: Let $G = P_n, n \geq 2$. Let G' be the graph obtained by duplicating all the edges of G by vertices.

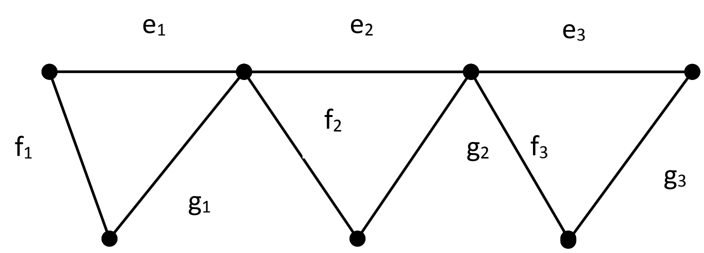
Case (1): Let $G = P_2$. The graph G' is given in the following figure. Since G' = C_3, $\gamma_{se}'(G') = 1$.



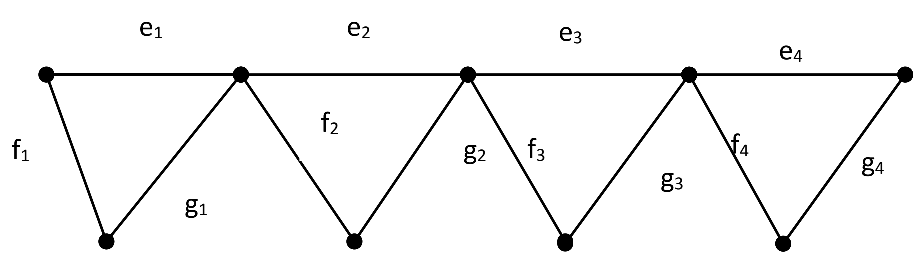
Case (2): Let $G = P_3$. The graph G' is given in the following figure. Then S₁= {e₁, g₂}, S₂= {e₂, f₁}, S₃= {g₁, g₂}, S₄= {f₁, f₂} are the strong efficient edge dominating sets of G' and $|S_i| = 2$, $1 \le i \le 4$. Therefore $\gamma'_{se}(G') \le 2$ Also no other strong efficient edge dominating sets of G' exists. Hence $\gamma'_{se}(G') = 2$.



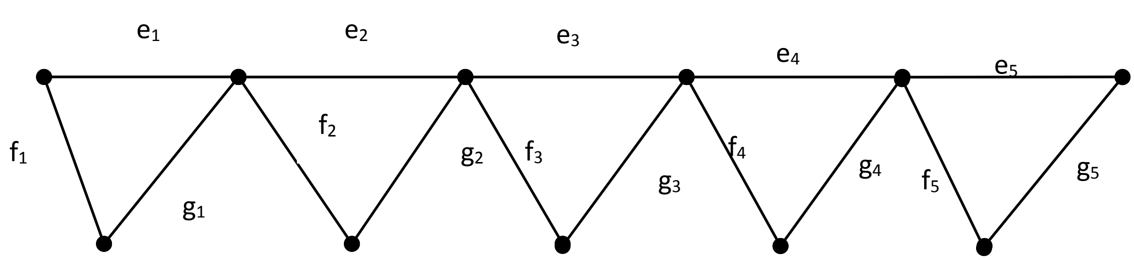
Case (3): Let $G = P_4$. The graph G' is given in the following figure. Then S= {e_2, f_1, g_3} is the unique strong efficient edge dominating sets of G' and $|S| = 3$. Therefore $\gamma'_{se}(G') = 3$.



Case (4): Let $G = P_5$. The graph G' is given in the following figure. Then S_1= {e_2, f_1, e_4}, S_2= {e_3, g_1, g_4} are the strong efficient edge dominating sets of G' and $|S_i| = 3$, i = 1, 2. Therefore $\gamma'_{se}(G') \le 3$. Since $3 = \gamma'_{s}(G') \le \gamma'_{se}(G')$. Therefore $\gamma'_{se}(G') \ge 3$. Hence $\gamma'_{se}(G') = 3$.



Case (5): Let $G = P_6$. The graph G' is given in the following figure. Then S= {e_1, e_3, e_5} is the unique strong efficient edge dominating sets of G' and $|S| = 3$. Therefore $\gamma'_{se}(G') = 3$.



Conversely: Case (1): Let $G = P_{3n},\ n \geq 3$. Then $G^{'}$ has no strong efficient edge dominating set.

Proof: Let $G = P_{3n},\ n \geq 3$. Let V(G$^{'}$) = {v_i / 1 ≤ i ≤ 3n, u_i / 1 ≤ i ≤ 3n-1}, $e_i = v_i v_{i+1}$, 1 ≤ i ≤ 3n – 1, $f_i = u_iv_i$, $g_i = u_i v_{i+1}$, 1 ≤ i ≤ 3n - 1. Then E(G') = {e_i, f_i, g_i / 1 ≤ i ≤ 3n - 1}. Deg e_1 = deg e_{3n-1} = 4, deg e_i = 6, 2 ≤ i ≤ 3n – 2, deg f_1 = deg g_{3n-1} = 2and the remaining edges have degree 4. Let S be a strong efficient edge dominating set of G'. Since deg e_i = 6 = $\Delta(G^{'})$, 2 ≤ i ≤ 3n – 2, either the edge e_2 or e_3 belongs to S.

Sub case 1(a): Suppose the edge e_2 belongs to S. Then it strongly efficiently dominates e_1, e_3, f_2, f_3, g_1, g_2. Also the edges e_5, e_8, e_{11},…,e_{3n-4} belong to S. If the edge e_{3n-2} belongs to S then $|N_S[e_{3n-3}] \cap S| = |\{e_{3n-2}, e_{3n-4}\}| = 2 > 1$, a contradiction. Hence there is no edge in S to strongly efficiently dominate e_{3n-2}. Therefore S is not a strong efficient edge dominating set of $G^{'}$. Hence $G^{'}$ has no strong efficient edge dominating set.

Sub case 1(b): Suppose the edge e_3 belongs to S. Then it strongly efficiently dominates e_2, e_4, f_3, f_4, g_2, g_3. Also the edges e_1, e_6, e_9, …, e_{3n-3}, e_{3n-1} belong to S. Also any one of the edges from each of the following set of edges $\{g_4, f_5\}, \{g_7, f_8\}, ..., \{g_{3n-5}, f_{3n-4}\}$ belongs to S. Suppose the edge g_4 belongs to S then $|N_S[f_4] \cap S| = |\{e_3, g_4\}| = 2 > 1$, a contradiction. Suppose the edge f_5 belongs to S then $|N_S[g_5] \cap S| = |\{e_6, f_5\}| = 2 > 1$, a contradiction. Hence there is no edge in S to strongly efficiently dominate g_4 and f_5.The proof is similar for the remaining set of edges. Therefore S is not a strong efficient edge dominating set of $G^{'}$. Hence $G^{'}$ has no strong efficient edge dominating set.

Case (2): Let $G = P_{3n+1},\ n \geq 2$. Then $G^{'}$ has no strong efficient edge dominating set.

Proof: Let $G = P_{3n+1},\ n \geq 2$. Let V(G$^{'}$) = {v_i / 1 ≤ i ≤ 3n+1, u_i / 1 ≤ i ≤ 3n}, $e_i = v_i v_{i+1}$, 1 ≤ i ≤ 3n, $f_i = u_i v_i$, $g_i = u_i v_{i+1}$, 1 ≤ i ≤ 3n. Then E(G') = {e_i, f_i, g_i / 1 ≤ i ≤ 3n}. Deg e_1 = deg e_{3n} = 4, deg e_i = 6, 2 ≤ i ≤ 3n – 1, deg f_1 = deg g_{3n} = 2 and the remaining edges have degree 4. Let S be a strong efficient edge dominating set of G'. Since deg e_i = 6 = $\Delta(G^{'})$, 2 ≤ i ≤ 3n –1, either the edge e_2 or e_3 belongs to S.

Sub case 2(a): Suppose the edge e_2 belongs to S. Then it strongly efficiently dominate e_1, e_3, f_2, f_3, g_1, g_2. Also the edges e_5, e_8, e_{11},…, e_{3n-1} belongs to S. Also any one of the edges from each of the following set of edges $\{g_3, f_4\}, \{g_6, f_7\}, ..., \{g_{3n-3}, f_{3n-2}\}$ belongs to S. If the edge g_3 belongs to S then $|N_S[f_3] \cap S| = |\{e_2, g_3\}|$ =2 > 1, a contradiction. Suppose the edge f_4 belongs to S then $|N_S[g_4] \cap S| = |\{e_5, f_4\}| = 2 > 1$, a contradiction. Hence there is no edge in S to strongly efficiently dominate g_3 and f_4. The proof is similar for the remaining set of edges. Therefore S is not a strong efficient edge dominating set of $G^{'}$. Hence $G^{'}$ has no strong efficient edge dominating set.

Sub case 2(b): Suppose the edge e_3 belongs to S. Then it strongly efficiently dominates e_2, e_4, f_3, f_4, g_2, g_3. Also the edges e_1, e_6, e_9, …, e_{3n} belongs to S. Also any one of the edges from each of the following set of edges $\{g_4, f_5\}, \{g_7, f_8\}, ..., \{g_{3n-2}, f_{3n-1}\}$ belongs to S. If the edge $g_4$belongs to S then $|N_S[f_4] \cap S| = |\{e_3, g_4\}| = 2 > 1$, a contradiction. If the edge f_5 belongs to S then $|N_S[g_5] \cap S| = |\{e_6, f_5\}| = 2 > 1$, a contradiction.Hence there is no edge in S to strongly efficiently dominate g_4 and f_5. The proof is similar for the remaining set of edges. Therefore S is not a strong efficient edge dominating set of $G^{'}$. Hence $G^{'}$ has no strong efficient edge dominating set.

Case (3): Let $G = P_{3n+2}$, $n \geq 2$. Then $G^{'}$ has no strong efficient edge dominating set.

Proof: Let $G = P_{3n+2}$, $n \geq 2$. Let V(G') = {v_i / 1 ≤ i ≤ 3n+2, u_i / 1 ≤ i ≤ 3n+1},,e_i = v_i v_{i+1}, f_i = u_iv_i, g_i = u_i v_{i+1}, 1 ≤ i ≤ 3n+1. Then E(G') = {e_i, f_i, g_i / 1 ≤ i ≤ 3n+1}. Deg e_1 = deg e_{3n+1} = 4, deg e_i = 6, 2 ≤ i ≤ 3n, deg f_1 = deg g_{3n+1} = 2and the remaining edges have degree 4. Let S be a strong efficient edge dominating set of G'. Since deg e_i = 6 = $\Delta(G^{'})$, 2 ≤ i ≤ 3n, either the edge e_2 or e_3 belongs to S.

Sub case 3(a): Suppose the edge e_2 belongs to S. Then it strongly efficiently dominates e_1, e_3, f_2, f_3, g_1, g_2. Also the edges e_5, e_8, e_{11},…, e_{3n-1}, e_{3n+1} belongs to S. Also any one of the edges from each of the following set of edges $\{g_3, f_4\}, \{g_6, f_7\}, ..., \{g_{3n-3}, f_{3n-2}\}$ belongs to S. If the edge g_3 belongs to S then, $|N_S[f_3] \cap S| = |\{e_2, g_3\}|$ =2 > 1, a contradiction. Suppose the edge f_4 belongs to S then $|N_S[g_4] \cap S| = |\{e_5, f_4\}| = 2 > 1$, a contradiction. Hence there is no edge in S to strongly efficiently dominate g_3 and f_4. The proof is similar for the remaining set of edges. Therefore S is not a strong efficient edge dominating set of $G^{'}$. Hence $G^{'}$ has no strong efficient edge dominating set.

Sub case 3(b): Suppose the edge e_3 belongs to S. Then it strongly efficiently dominates e_2, e_4, f_3, f_4, g_2, g_3. Also the edges e_1, e_6, e_9, …, e_{3n} belongs to S. Also any one of the edges from each of the following set of edges $\{g_4, f_5\}, \{g_7, f_8\}, ..., \{g_{3n-2}, f_{3n-1}\}$ belongs to S. If the edge g_4 belongs to S then $|N_S[f_4] \cap S| = |\{e_3, g_4\}| = 2 > 1$, a contradiction. If the edge f_5 belongs to S then $|N_S[g_5] \cap S| = |\{e_6, f_5\}| = 2 > 1$, a contradiction. Hence there is no edge in S to strongly efficiently dominate g_4 and f_5. The proof is similar for the remaining set of edges. Therefore S is not a strong efficient edge dominating set of $G^{'}$. Hence $G^{'}$ has no strong efficient edge dominating set.

Theorem 2.7: Let $G = K_{1,n}, n \geq 1$. Let G' be the graph obtained by duplicating all the edges of G by vertices. Then $\gamma_{se}^{'}(G') = n, n \geq 1$.

Proof: Let $G = K_{1,n}, n \geq 1$. Let V (G') = {v, v_i, u_i / 1 ≤ i ≤ n, $n \geq 1$} and let e_i = vv_i, f_i= vu_i , g_i = u_i v_i. Then E (G') = {e_i, f_i, g_i, 1 ≤ i ≤ n, $n \geq 1$}. Then deg e_i = deg f_i = 2n, deg g_i = 2, 1 ≤ i ≤ n. Let S be a strong efficient edge dominating set of G'. Since deg e_i = deg f_i= 2n = $\Delta(G^{'})$, 1 ≤ i ≤ n, any one of the edge e_i or $f_i$must belongs to S. Without loss of generality, suppose the edge e_1 belongs to S. Then it strongly efficiently dominate all the edges e_i, 2 ≤ i ≤ n and f_i, 1 ≤ i ≤ n and the edge g_1. Also the edges g_i, 2 ≤ i ≤ n must belong to S. Therefore S = {e_1, g_i / 2 ≤ i ≤ n} is a strong efficient edge dominating set of $G^{'}$ and |S| = n, $n \geq 1$. Therefore $\gamma_{se}^{'}(G^{'}) \leq n, n \geq 1$. The proof is similar if the edges e_i, 2 ≤ i ≤ n or f_i, 1 ≤ i ≤ n belong to S. Also no set with less than n edges is a strong efficient edge dominating set of $G^{'}$. Therefore $\gamma_{se}^{'}(G^{'}) \geq n, n \geq 1$. Hence $\gamma_{se}^{'}(G^{'}) = n, n \geq 1$.

Theorem 2.8: Let $G = D_{r,s}, r, s \geq 1$. Let G' be the graph obtained by duplicating all the edges of G by vertices. Then $\gamma_{se}^{'}(G') = r + s + 1, r, s \geq 1$.

Proof: Let $G = D_{r,s}, r, s \geq 1$. Let V(G') = {u, v, u_i, v_j, u_i', v_j' / 1 ≤ i ≤ r, 1 ≤ j ≤ s, r, s ≥ 1} and let e = uv, f = uw, g = vw, e_i = uu_i, e_i' = uu_i',f_j = vv_j, f_j' = vv_j',g_i = u_iu_i',h_j = v_jv_j', 1 ≤ i ≤ r, 1 ≤ j ≤ s, r, s ≥ 1. Then E(G') = {e, f, g, e_i,e_i', f_i, f_j ', g_i, h_j/1 ≤ i ≤ r, 1 ≤ j ≤ s, r, s ≥ 1}. Deg e = 2(r+s+1), deg f = 2r, deg g = 2s, deg e_i = deg e_i'= 2r, deg f_j = deg f_j'= 2s, deg g_i = deg h_j = 2, 1 ≤ i ≤ r, 1 ≤ j ≤ s, r, s ≥ 1. Since deg e = 2(r+s+1) = $\Delta(G^{'})$, 1 ≤ i ≤ r, 1 ≤ j ≤ s, r, s ≥ 1, S = {e, g_i, h_j / 1 ≤ i ≤ r, 1 ≤ j ≤ s, r, s ≥ 1} is the unique strong efficient edge dominating set of G' and |S| = r+s+1, 1 ≤ i ≤ r, 1 ≤ j ≤ s, r, s ≥ 1. Hence $\gamma'_{se}(G') = r + s + 1,\ \ r, s \geq 1.$

Theorem 2.9: Let $G = W_{3n}, n \geq 2$. Let G' be the graph obtained by duplicating all the edges of G by vertices. Then $\gamma'_{se}(G') = 2n,\ n \geq 2.$

Proof: Let $G = W_{3n}, n \geq 2$. Let V(G') = {u,v_i, u_i, w_i / 1 ≤ i ≤ 3n - 1} and $e_i = v_iv_{i+1}$, 1 ≤ i ≤ 3n-2, $e_{3n-1} = v_{3n-1}v_1$ $e_i' = vv_i$, 1 ≤ i ≤ 3n-1, $f_i = u_i v_i$, 1 ≤ i ≤ 3n-1, $g_i = u_i v_{i+1}$, 1 ≤ i ≤ 3n-2, $g_{3n-1} = u_{3n-1} v_1$, , $f_i' = v_iw_i$,$g_i' = vw_i$,1 ≤ i ≤ 3n-1. Then E (G') = {e_i, e_i', f_i ,f_i', g_i ,g_i'/ 1 ≤ i ≤ 3n -1}. Deg e_i = 10, deg e_i'= 6n+2, deg f_i = deg f_i' = deg g_i = 6, deg g_i' = 6n – 2, 1 ≤ i ≤ 3n -1. The edges e_i' are adjacent with each other. To dominate them, any one e_i' is considered. Without loss of generality, let it be e_1'. S = {e_1', e_3, e_6, …, e_{3n-3},f_2' , f_2', f_5' ,…, f_{3n-1}'} is astrong efficient edge dominating set of G' and |S| =2n, n ≥ 2. Therefore $\gamma'_{se}(G') \leq 2n, n \geq 2$. The proof is similar if the edges e_i', 2 ≤ i ≤ n belong to S. Also no set with less than 2n edges is a strong efficient edge dominating set of G'. Therefore $\gamma'_{se}(G') \geq 2n,\ n \geq 2$. Hence $\gamma'_{se}(G') = 2n,\ n \geq 2.$

Theorem 2.10: Let $G = C_{3n}, n \geq 1$. Let G' be the graph obtained by duplicating all the edges of G by vertices.. Then G' has no strong efficient edge dominating set.

Proof: Let G = C_{3n}, $n \geq 1$. Let V(G') = {u_i, v_i / 1 ≤ i ≤ 3n}, $e_i = v_i v_{i+1}$, 1 ≤ i ≤ 3n-1, $e_{3n} = v_{3n} v_1$, $f_i = u_i v_i$, 1 ≤ i ≤ 3n, $g_i = u_iv_{i+1}$, 1 ≤ i ≤ 3n-1, $g_{3n} = u_{3n}v_1$. Then E(G') = {e_i, f_i ,g_i / 1 ≤ i ≤ 3n}. Deg e_i = 6, 1 ≤ i ≤ 3n, deg f_i = deg g_i = 4, 1 ≤ i ≤ 3n. Let S be a strong efficient edge dominating set of G'. Since deg e_i = 6 = $\Delta(G')$, 1 ≤ i ≤ 3n, any one of the edges e_i,1 ≤ i ≤ 3n belongs to S. Without loss of generality let it be the edge e_1.Then it strongly efficiently dominates e_2, e_{3n}, f_1, f_2, g_1,g_{3n}. Also the edges e_4, e_7, e_{10}, …, e_{3n-2} belongs to S. Also any one of the edges from each of the following set of edges $\{g_2, f_3\}, \{g_5, f_6\}, ..., \{g_{3n-1}, f_{3n}\}$ must belong to S. If the edge g_2 belongs to S then $|N_S[f_2] \cap S| = |\{e_1, g_2\}| = 2 > 1$, a contradiction. If the edge f_3 belongs to S then $|N_S[g_3] \cap S| = |\{e_4, f_3\}| = 2 > 1$, a contradiction. Hence there is no edge in S to strongly efficiently dominate g_2 andf_3. The proof is similar for the remaining set of edges. Therefore S is not a strong efficient edge dominating set of G'. Hence G' has no strong efficient edge dominating set.

III. CONCLUSION

In this paper, the strong efficient edge domination number of some graphs obtained by duplicating their elements is determined.

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